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Global stability of time-dependent flows: impulsively heated or cooled fluid layers

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The method of energy is used to discuss the stability of time-dependent diffusive temperature profiles in fluid layers subject to impulsive changes in surface temperature.

Bounds for the ratio of disturbance energy production to dissipation are found to be parametric functions of time because the basic temperature develops through diffusion. This time dependence leads to the demarcation of regions of stability in a Rayleigh number–time plane and the interpretation of these regions is given. Numerical results are presented for the cases of impulsive heating and cooling of initially isothermal fluid layers. New global stability results which give the Rayleigh number below which the diffusive solution to the Boussinesq equations is unique are reported for these cases.

1. Introduction

The stability of flows whose base state is time-dependent has received increased attention in recent years. The problems considered thus far fall into two main categories.

Considerable theoretical work has been done on the linear stability of harmonically modulated base flows, exemplified by Yih & Li (1972), Gresho & Sani (1970), Venezian (1969), and Rosenblat and co-workers (Rosenblat & Herbert 1970; Rosenblat & Tanaka 1971). Within the confines of linear theory, the approach usually taken in these works is to remove the spatial dependence in the governing equations by Fourier decomposition, followed by a Galerkin expansion of the disturbance variables in the remaining spatial direction in terms of trial functions with time-dependent amplitudes. In some cases, a long-wave expansion is possible (Yih 1968). The long-time behaviour of the resulting linear amplitude equations is then determined by a straightforward application of Floquet theory. Given the convergence of the Galerkin expansions, such an approach is rigorous and the results obtained to date can be assumed to describe properly the stability of such systems with respect to infinitesimal disturbances.

The situation in the case of base states which evolve in time following and impulsive step change is far less satisfying. The Galerkin equations describing the initial growth of small disturbances are linear but non-autonomous, and one loses the property of translation to zero time, which applies for steady flows, i.e. disturbances can no longer be set proportional to $\exp(\sigma t)$. The possibility of obtaining the asymptotic time dependence is also lost, since no periodicity is present in the base state. Early approaches to the problem consisted of 'freezing' the base state at a given time and determining its marginal stability; the time thus appears only parametrically (Lick 1965; Currie 1967). This attack proceeds from the assumption that disturbances are growing faster (in some suitable sense) than the base state is evolving; it is obvious, however, that the worst time to apply this assumption is at marginal conditions. Onset times for instability predicted by this approach are generally quite low. By onset time, we mean the time at which the disturbances are observable by a suitable transducer. This time depends to some extent upon the sensitivity of the measuring device, but different methods of observation seem to yield the same value. 'Momentary instability' methods have also been attempted with little success and have been critically evaluated by Gresho & Sani (1971); see also Chen & Kirchner (1971). An alternative approach to this class of problems has been taken by Foster (1965, 1968) followed by Mahler, Schechter & Wissler (1968), Mahler & Schechter (1970), Gresho & Sani (1971) and Chen & Kirchner (1971). The linear (Galerkin) equations are integrated numerically subject to some initial conditions and the growth of disturbances is followed as a function of time. Two difficulties arise in this approach. The first is conceptual in nature and involves a subjective decision as to how large disturbances must grow before they become observable. The problem occurs because the stability equations in this case are linear and homogeneous and hence disturbance amplitudes are indeterminate; only amplifications *relative to the initial conditions* can be computed. The occurrence of visible motions corresponding to the loss of stability must then be correlated with the computed amplification through comparison with experiment. Unfortunately no single value of the amplification will predict the experimentally observed onset times. The experiments of Blair & Quinn (1969), Mahler & Schechter (1970), Chen & Kirchner (1971) and Davenport & King (1972) indicate that the empirically determined value of the amplification of kinetic energy corresponding to onset lies between 10 and 10^8. A second difficulty with this class of methods is the surprising effect the choice of initial conditions has on the computed amplifications (Gresho & Sani 1971, figure 5). The strong dependence does not appear to reflect properly the behaviour observed experimentally and is at odds with one's physical intuition that initial conditions should be relatively unimportant.

Resolution of these crucial problems would thus seem to await the development of a nonlinear theory. A first step in this direction has been taken by Davis (1971), who has extended the Stuart–Watson method to time-varying base states. The method remains fairly formal however since in most cases the complexity of the results of linear theory makes the possibility of major analytical progress seem remote.

The objective of the present paper is to determine sufficient conditions for stability of time-dependent motions using the method of energy. It has been dramatically demonstrated in recent years that the energy method is complementary to linear stability theory, for the two taken together clearly limit the regions in parameter space for which subcritical instabilities are possible; see Joseph (1971). It thus seems appropriate that, in parallel with the linear stability work outlined above, the development of the energy theory for time-dependent base states should begin. In this spirit, the present paper seeks to clarify the early work of Conrad & Criminale (1965*a*, *b*) and amplify some results which are latent in the work of Joseph (1966) and Joseph & Shir (1966). We do so within the context of that class of problems for which rigorousst ability results *of any sort* are sorely lacking, namely the stability of fluid layers subject to impulsive heating or cooling. Davis (1972) has applied the energy method to this problem in a limited context. By treating the time-dependent part of the developing base-state temperature field *as a disturbance*, simple application of Joseph's (1966) results yields the somewhat obvious results that layers subject to such heating histories are asymptotically stable for Rayleigh numbers below the classical critical value. Such a result may be deduced by purely physical reasoning. If the layer is to be convectively unstable for all time, then the energy necessary to drive the motions must be derived solely from the potential energy available owing to the (constant) imposed temperature difference. Clearly, this can be the case only for supercritical Rayleigh numbers. The results developed in the present paper will be seen to complement this result, giving a definite value of the time at which disturbances, if present at subcritical Rayleigh numbers, must begin to decrease strongly.

In §2 we develop the energy evolution equations for such base states. The optimal stability boundary is discussed in §3 as well as the qualitative shape of the boundary for small and large times. Section 4 presents computations for three cases of interest. As a result of these calculations new global stability results are now available for fluid layers subject to rapid heating or cooling.

2. The energy identities

We consider a class of problems in which an initially quiescent fluid layer which may be stably or unstably stratified (but linearly stable) is subjected to impulsive changes in the thermal conditions at the bounding horizontal surfaces. Let the fluid depth be d, the thermal diffusivity κ and a characteristic (constant) temperature difference ΔT. Then under the scaling $\{\mathbf{r}, t, \theta\} = \{d, d^2/\kappa, \Delta T\}$ a diffusive base state exists in which the fluid velocity $\mathbf{v} \equiv 0$ and the dimensionless temperature $\overline{\theta}$ satisfies the diffusion equation

$$\partial\overline{\theta}/\partial t = \partial^2\overline{\theta}/\partial z^2 \tag{2.1}$$

subject to appropriate initial and boundary conditions.

We wish to inquire under what conditions this base state remains stable to disturbances of arbitrary amplitude. The derivation of the energy indentities closely parallels that of Joseph (1966), and hence only the essential steps are given in detail here.

Let the scalings of disturbance variables be

$$\{\mathbf{r}, t, \mathbf{v}, p, \theta\} = \{d, d^2/\kappa, \kappa/d, \rho\kappa\nu/d^2, \Delta T\}.$$

The nonlinear Boussinesq equations governing these disturbances are then given in dimensionless form by

$$\sigma^{-1}(\partial \mathbf{v}/\partial t + \mathbf{v}.\nabla\mathbf{v}) = -\nabla p + \nabla^2\mathbf{v} + Ra\,\theta\mathbf{k}, \tag{2.2a}$$

$$\partial\theta/\partial t + \mathbf{v}.\nabla\theta = \nabla^2\theta - w(\partial\bar{\theta}/\partial z). \tag{2.2b}$$

The notation is standard. σ is the Prandtl number, $Ra = g\beta\Delta T d^3/\kappa\nu$ is the Rayleigh number, $\mathbf{k}$ the unit vector in the z direction (z is measured positive upwards) and w the vertical component of velocity. For non-deformable bounding surfaces across which there is no flow, and at which a thermal condition of the third kind holds, i.e. $\mathbf{n}.\nabla\theta = -L\theta$, one can derive the energy identities from (2.2) in the usual manner (Joseph 1966, p. 164):

$$\frac{1}{2}\frac{dK(\mathbf{v})}{dt} = \frac{1}{2\sigma}\frac{d}{dt}\langle|\mathbf{v}|^2\rangle = -\langle\nabla\mathbf{v}:\nabla\mathbf{v}\rangle + Ra\langle w\theta\rangle, \tag{2.3a}$$

$$\frac{1}{2}\frac{d\Theta(\theta)}{dt} = \frac{1}{2}\frac{d}{dt}\langle\theta^2\rangle = -\langle|\nabla\theta|^2\rangle - \left\langle\theta w\frac{\partial\bar{\theta}}{\partial z}\right\rangle - \oint L\theta^2. \tag{2.3b}$$

We then take a linear combination of (2.3a, b) so as to form a positive-definite 'energy' functional $E' = K + \lambda Ra\Theta$; λ, $Ra > 0$. Thus,

$$\frac{1}{2}\frac{dE'}{dt} = Ra\left(\langle w\theta\rangle - \lambda\left\langle w\theta\frac{\partial\bar{\theta}}{\partial z}\right\rangle\right) - \left[\langle\nabla\mathbf{v}:\nabla\mathbf{v} + \lambda Ra|\nabla\theta|^2\rangle + Ra\,\lambda\oint L\theta^2\right]. \tag{2.4}$$

Equation (2·4) may be put into symmetric form by introducing

$$\phi = (\lambda Ra)^{\frac{1}{2}}\theta \quad \text{and} \quad E = K(\mathbf{v}) + \Theta(\phi).$$

The resulting energy identity may then be shown to satisfy the differential inequality

$$D^{-1}\frac{1}{2}\frac{dE}{dt} \leqslant -1 + \frac{Ra^{\frac{1}{2}}}{\rho_\lambda}, \tag{2.5}$$

where $D \equiv \langle\nabla\mathbf{v}:\nabla\mathbf{v} + |\nabla\phi|^2\rangle + \oint L\phi^2$ and ρ_λ is a solution to the maximum problem

$$\frac{1}{\rho_\lambda} = \max_h\left\{\frac{\langle w\phi\rangle}{\lambda^{\frac{1}{2}}} - \lambda^{\frac{1}{2}}\left\langle w\phi\frac{\partial\bar{\theta}}{\partial z}\right\rangle\right\}, \tag{2.6a}$$

$$D = 1. \tag{2.6b}$$

The space h over which the maximum is taken is discussed in detail by Davis (1969). It is sufficient to note that it is comprised of couples $(\mathbf{v}, \phi)$ satisfying the thermal and kinematic conditions at the boundaries together with the constraint that $\mathbf{v}$ be solenoidal.

Equation (2.5) thus forms the starting point for our discussion. Clearly the disturbance energy decreases whenever $Ra^{\frac{1}{2}} < \rho_\lambda$. However, ρ_λ depends parametrically upon the dimensionless time t *via* the time dependence of the base-state temperature gradient. Thus solutions of the maximum problem for varying t will produce a curve in the ρ_λ, t plane. It is from this curve that we may draw conclusions regarding the stability of the flow.

In particular, our main interest is in proving strong stability in the sense of

Joseph (1971), that is, exponential decay of the energy with time. To develop this property in full, we may further reduce (2.5) to

$$\tfrac{1}{2}(dE/dt) \leqslant E\xi^2(-1+Ra^{\frac{1}{2}}/\rho_\lambda(t)) \tag{2.7}$$

whenever $Ra < \rho_\lambda(t)$. Here ξ^2 is the usual decay constant whose value is available for plane layers (Joseph 1965). The validity of (2.7) relies both upon the use of the isoperimetric inequality $D \geqslant \xi^2 E$ and the requirement $Ra^{\frac{1}{2}} < \rho_\lambda(t)$. The questions of lower bounds for onset times and global stability clearly depend upon the shape of the stability boundary in the ρ_λ, t plane, but it is convenient to postpone a discussion of these points until the nature of the optimal stability boundary has been elucidated.

3. The optimal stability boundary

As with similar problems involving both vector and scalar fields, we wish to choose the 'coupling parameter' λ so that the region of stability is as large as possible. In the present problem, therefore, we inquire into the consequences of allowing λ to vary so as to yield the maximum value of ρ_λ. Thus in addition to (2.6a, b) we solve the problem

$$\tilde{\rho}(t) = \max_{\lambda} \rho_\lambda(\lambda, t). \tag{3.1}$$

This process may then be repeated for different values of the time, yielding an optimal stability boundary $\tilde{\rho}(t)$ and a corresponding set of coupling parameters $\tilde{\lambda} = \lambda_{\text{opt}}(t)$. However, care must be taken at this point since in deriving (2.4) from (2.3a, b) λ was assumed constant. The difficulty is one of ensuring that the time dependence of $\tilde{\lambda}$ does not preclude the existence of a suitable Lyapunov function (or functional) which is bounded by E' and hence bounded by the right-hand side of (2.5). The simplest criterion available (and one which is shown below to pose a serious constraint for the present class of base states) is the requirement that, in addition to $\lambda > 0$, we also must have $d\lambda/dt \leqslant 0$.

If these constraints are met, the result follows immediately from the inequalities

$$\frac{1}{2}\frac{d}{dt}(\langle|\mathbf{v}|^2/\sigma + \lambda\, Ra\, \theta^2\rangle) = \frac{1}{2}\left(\frac{dK}{dt} + \lambda\, Ra\frac{d\Theta}{dt}\right) + \tfrac{1}{2}Ra\,\Theta\lambda' \leqslant \frac{1}{2}\frac{dE'}{dt} \leqslant \ldots . \tag{3.2}$$

It is argued below that the additional constraint that $\tilde{\lambda}$ be monotone decreasing is probably an active one for the class for base states considered here, and the numerical calculations detailed in §4 bear this out. We show below, however, that, under relatively mild restrictions on the function $\rho_\lambda(t)$, the optimal stability boundary computed via (3.1) is in fact the correct one.

In order to establish properties regarding the shape of the optimal stability boundary and the parametric time dependence of $\tilde{\lambda}$, it is convenient to consider the Euler–Lagrange equations for problem (2.6a). Introducing multipliers for the constraint (2.6b) and the solenoidal constraint on $\mathbf{v}$, we obtain the following Euler–Lagrange equations in the usual manner:

$$\nabla . \mathbf{v} = 0, \tag{3.3a}$$

$$\frac{\rho_\lambda}{2}\left(\frac{1}{\lambda^{\frac{1}{2}}}-\lambda^{\frac{1}{2}}\frac{\partial\bar{\theta}}{\partial z}\right)w+\nabla^2\phi = 0, \tag{3.3b}$$

$$\frac{\rho_\lambda}{2}\left(\frac{1}{\lambda^{\frac{1}{2}}}-\lambda^{\frac{1}{2}}\frac{\partial\bar{\theta}}{\partial z}\right)\phi\mathbf{k}+\nabla^2\mathbf{v}-\nabla\pi = 0. \tag{3.3c}$$

By parametric differentiation, one finds (Jospeh & Shir 1966)

$$\left\langle\left[\delta\lambda\left(\frac{\partial\bar{\theta}}{\partial z}-1\right)+\lambda\delta\left(\frac{\partial\bar{\theta}}{\partial z}\right)\right]w\phi\right\rangle = -\delta(\lambda^{\frac{1}{2}}/\rho_\lambda). \tag{3.4}$$

Equation (3.4) can be used to deduce two important properties necessary for the discussion. The first of these is that if $(\mathbf{v},\phi)$ are maximizing functions at any time t, then the best λ, i.e. the one for which $\partial\rho_\lambda/\partial\lambda = 0$, is given by

$$\tilde{\lambda} = \frac{-\langle w\phi\rangle}{\langle w\phi(\partial\bar{\theta}/\partial z)\rangle}. \tag{3.5}$$

This is a specialization of a result due to Joseph & Shir (1966). Now for base-state temperature gradients resulting from impulsive changes in boundary temperatures, $\partial\bar{\theta}/\partial z$ will in general be given by a steady part plus a transient term which decays rapidly in time for moderate times. Equation (3·5) thus suggests that the additional constraint $d\tilde{\lambda}/dt \leqslant 0$ might not be satisfied by the optimal coupling parameter.

The second property of interest concerns the shape of the optimal stability boundary in the limit of large or small times. If λ is chosen as its best value, and $(\mathbf{v},\phi)$ is a maximizing couple, then

$$d\tilde{\rho}/dt = (\partial\tilde{\rho}/\partial t)_\lambda \tag{3.6}$$

and thus from (3.4) we have

$$\frac{\tilde{\lambda}^{\frac{1}{2}}}{\tilde{\rho}^2}\frac{d\tilde{\rho}}{dt} = \tilde{\lambda}\left\langle\frac{\partial^2\bar{\theta}}{\partial z\,\partial t}w\phi\right\rangle. \tag{3.7}$$

Now from the original maximum problem, (2.6a) we have the additional formula

$$\tilde{\lambda}^{\frac{1}{2}}/\tilde{\rho} = -\langle(\tilde{\lambda}(\partial\bar{\theta}/\partial z)-1)\,w\theta\rangle. \tag{3.8}$$

Combining (3.7) and (3.8) and using (3.5) we have

$$\frac{1}{\tilde{\rho}}\frac{d\tilde{\rho}}{dt} = -\frac{\tilde{\lambda}}{2}\left\langle\frac{\partial^2\bar{\theta}}{\partial z\,\partial t}w\phi\right\rangle\bigg/\langle w\phi\rangle. \tag{3.9}$$

Further integration of (3.9) would seem to require detailed knowledge of both $\tilde{\lambda}$ and the maximizing functions (w,ϕ). However, it is obvious that as $t\to\infty$, $\partial\bar{\theta}/\partial t\to 0$, $\tilde{\lambda}\to\tilde{\lambda}^0$ and $\tilde{\rho}$ approaches its constant limiting value. Furthermore, under the assumption that $\lim_{t\to 0}\tilde{\lambda}(t)$ exists, we have

$$\lim_{t\to 0}\frac{1}{\tilde{\rho}}\frac{d\tilde{\rho}}{dt} = -\frac{\tilde{\lambda}^0}{2}\lim_{t\to 0}\left\{\left\langle\frac{\partial^2\bar{\theta}}{\partial z\,\partial t}w\phi\right\rangle\bigg/\langle w\phi\rangle\right\}. \tag{3.10}$$

We turn now to an interpretation of the stability boundary. Clearly two possibilities present themselves, depending upon the slope of the optimal stability boundary in the $\tilde{\rho}$, t plane.

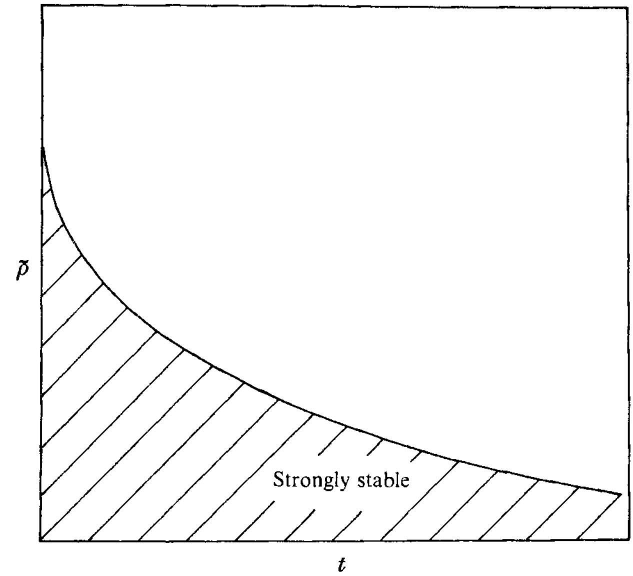

FIGURE 1. Stability boundary with negative slope. The line indicates the solution to problem (2.6) and (3.1). In the shaded area, the energy is guaranteed to decrease exponentially.

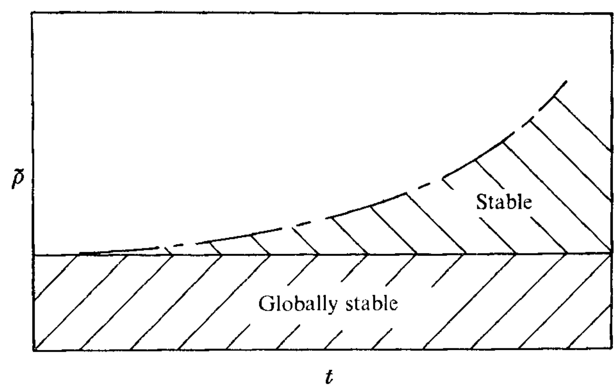

FIGURE 2. Stability boundary with positive slope. Disturbances decay exponentially for all $(Ra^{\frac{1}{2}}, t)$ to the right of the curve. No statement can be made concerning the region to the left of the curve.

Figure 1 depicts the case in which the slope is negative, e.g. as in Conrad & Criminale (1965*a*). The curve represents the locus of points $(\tilde{\rho}, t)$. For any given Rayleigh number, the results yield lower bounds for the onset time for disturbances of arbitrary initial amplitude. For $(Ra^{\frac{1}{2}}, t)$ pairs to the left of the curve, the energy is guaranteed to decrease exponentially in accord with (2.7). The stability of the area lying to the right of the curve cannot be decided by the present method.

For stability boundaries with positive slopes (figure 2), no results regarding the onset time are available. However, everywhere to the right of the dotted curve (2.7) guarantees that disturbances must decay exponentially; at small

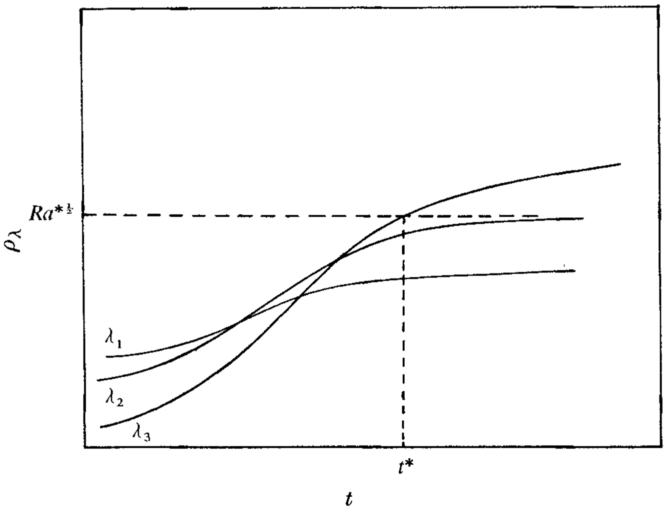

FIGURE 3. The function $\rho_\lambda(t)$ and the trajectory of an experiment in the ρ_λ, t plane.

times, i.e., to the left of the curve, no statement can be made. The boundary thus gives the time after which disturbances, if present, must begin to decrease exponentially. In addition to these results, global stability limits for stability boundaries of this type may be deduced. From (2.7) we have

$$E^{-1}\frac{dE}{dt} \leqslant -\xi^2(1-Ra^{\frac{1}{2}}/\tilde{\rho}(t)) \leqslant -\xi^2(1-Ra^{\frac{1}{2}}/\min_t \tilde{\rho}(t)). \tag{3.11}$$

Integration yields

$$E(t) \leqslant E(0)\exp\{-\xi^2(1-Ra^{\frac{1}{2}}/\min_t \tilde{\rho}(t))\,t\}. \tag{3.12}$$

Thus the layer is globally stable for all $Ra^{\frac{1}{2}} < \min_t \tilde{\rho}$, and the diffusive solution $\overline{\theta}(z,t)$ of the Boussinesq equations is therefore unique.

It remains to demonstrate that the process of constructing the optimal stability boundary (3.1) remains valid if the constraint $d\tilde{\lambda}/dt \leqslant 0$ does not hold. We accomplish this under the mild assumption (which appears unprovable but computationally true with one minor exception) that *for a given* λ, $\rho_\lambda(t)$ is a monotonic function of t. Consider boundaries for which $\rho_\lambda(t)$ is monotone increasing. Then the family of curves will appear as sketched in figure 3. The proof hinges on the fact that any experiment corresponds to the trajectory consisting of a level horizontal line on such a sketch. Thus, to prove, for instance, strong stability after a time t^* for $Ra = Ra^*$, fix the value of λ for which

$$Ra^{*\frac{1}{2}} = \tilde{\rho} = \max_\lambda \rho_\lambda(t^*).$$

Then, provided $\rho_\lambda(t)$ is monotone, one makes the usual arguments following (2.7) for that fixed λ. Similar arguments hold for boundaries with negative slope (figure 1), and will not be given.

It is clear from figure 3 that the process of constructing the optimal boundary is actually the computation of the envelope of $\rho_\lambda(t)$ curves, which then delimits the region in $Ra^{\frac{1}{2}}, t$ space in which one can prove strong stability in the usual manner.

4. Results for fluid layers subjected to rapid heating or cooling

In this section we present the results of the energy method for an initially isothermal fluid layer which is impulsively heated from below or cooled from above. The solution for the diffusive base state for these conditions is given as

$$\bar{\theta}(z,t) = 1 - z - \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^k \frac{\sin(n\pi z)}{n}\, e^{-(n\pi)^2 t}, \tag{4.1a}$$

$$\frac{\partial\bar{\theta}}{\partial z} = -1 - 2\sum_{n=1}^{\infty}(-1)^k \cos(n\pi z)\, e^{-(n\pi)^2 t}. \tag{4.1b}$$

In (4.1), if k is set to zero, we have heating from below, and for $k = n$, cooling from above. The boundaries were considered to be conducting ($L = \infty$), and either rigid–rigid or rigid–free.

It is convenient to combine (3.3*a–c*) into two coupled equations for w and ϕ. These are found in the usual way to be

$$\frac{\rho_\lambda}{2}\left(\frac{1}{\lambda^{\frac{1}{2}}} - \lambda^{\frac{1}{2}}\frac{\partial\bar{\theta}}{\partial z}\right)\nabla_1^2\phi + \nabla^4 w = 0, \tag{4.2a}$$

$$\nabla^2\phi + \frac{\rho_\lambda}{2}\left(\frac{1}{\lambda^{\frac{1}{2}}} - \lambda^{\frac{1}{2}}\frac{\partial\bar{\theta}}{\partial z}\right) w = 0, \tag{4.2b}$$

subject to $\phi = w = 0$ at $z = 0, 1$ and either $\partial w/\partial z = 0$ at fixed horizontal boundaries or $\partial^2 w/\partial z^2 = 0$ at free horizontal boundaries. Here ∇_1^2 is the horizontal Laplacian. It is interesting to note that, with $\lambda = 1$, equations (4.2) are the symmetric part of the frozen-time equations (Gresho & Sani 1971, p. 208). Thus at large times the time-dependent part of the gradient constitutes a small perturbation of the case of steady linear stratification, and the energy and frozen-time predictions must necessarily be close.

Since (4.2) are cyclic in the two directions normal to the z co-ordinate, we can Fourier decompose (w, ϕ) into modes characterized by a single wavenumber α. We then treat ρ_λ as the lowest eigenvalue of the set (4.2) and seek $\tilde{\rho} = \max_{\lambda}\min_{\alpha}\rho_\lambda$. The results were generated extremely efficiently using the Rayleigh–Ritz procedure. Following Fourier decomposition, the z-dependent functions $\hat{w}$ and $\hat{\phi}$ are expanded in series of trial functions which satisfy the appropriate boundary conditions. We have used 'beam' functions and sines for $\hat{w}$ and $\hat{\phi}$ respectively with up to 15 terms in each expansion. If N terms are taken in each expansion the Rayleigh–Ritz method results in a $2N \times 2N$ algebraic eigenvalue problem. This problem is transformed to a $N \times N$ symmetric problem, the eigenvalues of which are then determined by an adaptation of a method due to Martin & Wilkinson (1968). Details of the computational procedure are available from the author upon request.

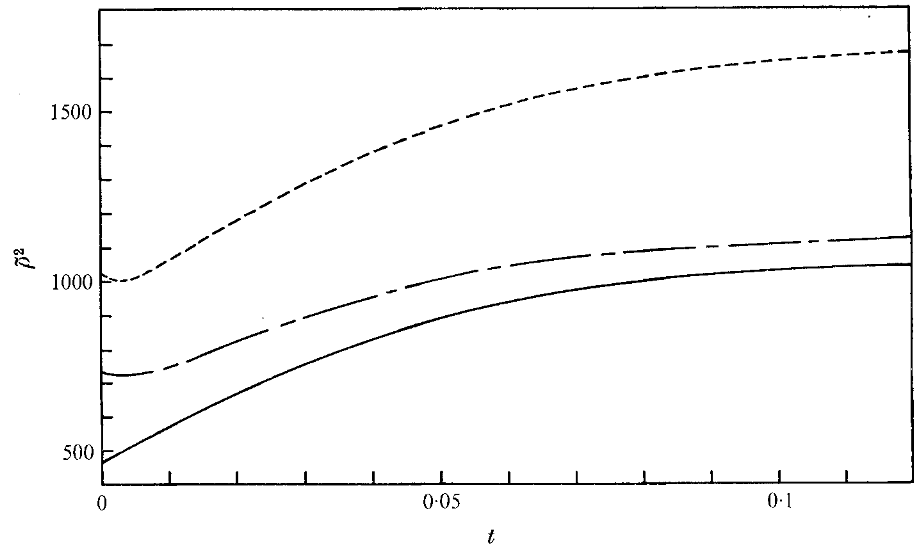

FIGURE 4. Computed results for initially isothermal layers. – – – –, fixed–fixed, heat or cool; — –, fixed–free, heat from below; ———, fixed–free, cool from above.

Conditions	$\rho^2_{\min}$
Rigid–rigid	996·6
Rigid–free–heat	718·6
Rigid–free–cool	465 (extrapolated)

TABLE 1. Global stability results

The results are shown in figure 4, and are believed to be accurate to five significant digits. As suggested by (3.5), the best λ was such that $d\tilde{\lambda}/dt \geqslant 0$ for all times. Furthermore the computed $\tilde{\lambda}$ appeared to approach a limiting value at small times.

For rigid–rigid conditions, the optimal stability boundaries for heating and cooling coincide owing to the obvious physical symmetry. This afforded a check on the numerical scheme. For the rigid–rigid and rigid–free–cool cases, the stability boundary has a distinct minimum at very short times. This was suspect at first, but after repeated independent checks, it was found to be genuine. It is also suggested by (3.10). At small times it can be shown that $\partial^2\overline{\theta}/\partial z\,\partial t$ is positive for all z and assuming the product $w\phi$ is single-valued on $0 \leqslant z \leqslant 1$, we have $\lim_{t\to 0} \tilde{\rho}^{-1} d\tilde{\rho}/dt < 0$. This minimum does not invalidate the arguments made at the end of the last section, since the small region to the left of the minimum is of no consequence in deducing either global stability or times after which disturbances must decay.

With regard to the long-time behaviour, it is known from the frozen-time results of Gresho & Sani (1971, figure 11) that at large times computed Rayleigh numbers are less than their limiting steady values. Since the energy and frozen-time results must be arbitrarily close as time increases, the trend of the curves in figure 4 at long times is substantiated.

Fluid layers impulsively heated or cooled are globally stable for Rayleigh numbers less than those given in table 1. For Rayleigh numbers between these global values and the linear Bénard limits, finite-amplitude convection cannot be excluded. The results however do furnish an explicit value for the time at which any convective motions, if present, must begin to decay exponentially.

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